Chaos and Exponentially Localized Eigenstates in Smooth Hamiltonian Systems

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We present numerical evidence to show that the wavefunctions of smooth classically chaotic Hamiltonian systems scarred by certain simple periodic orbits are exponentially localized in the space of unperturbed basis states. The degree of localization, as measured by the information entropy, is shown to be correlated with the local phase space structure around the scarring orbit; indicating sharp localization when the orbit undergoes a pitchfork bifurcation and loses stability.

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It is now well-known that classical periodic orbits have an enduring influence on the quantum mechanics and the semiclassics of classically chaotic quantum systems. The knowledge of all the isolated periodic orbits of such a system allows us to semiclassically estimate the eigenvalues through the application of Gutzwiller's trace formula [1]. Heller [2] has given theoretical arguments to show that for the classically chaotic quantum systems the eigenfunctions will show density enhancements, called scars, along the least unstable periodic orbits. Such localized probability density structures of the wavefunctions and their correspondence to the underlying periodic orbits are widely reported for many classically chaotic quantum systems, such as the hydrogen atom in a magnetic field [3]. The effect of localized states, in a quantum system with predominant classical chaos, has been experimentally observed using tunnel-current spectroscopy in semiconductor heterostructures [4]. The role played by the bifurcation properties of periodic orbits are also of considerable importance. The effect of orbit bifurcation is observed experimentally in the spectra of atoms in external fields [5,6].

The first numerical evidence for wavefunction localization in systems with classical chaos came from the studies of Bunimovich billiards [2,7]. For the kicked rotor, Grempel et. al. [8] have shown that the localization in momentum space is exponential, similar to Anderson localization in the case of charged particle dynamics in a series of potential wells with random depths. In smooth potentials like the coupled oscillators, adiabatic methods have been widely applied to predict some eigenvalues but the construction of adiabatic wavefunctions still remains an open problem. Polavieja et. al. [9] have used wavepacket propagation techniques to construct wavefunctions highly localized on a given classical periodic orbit. A qualitative study of the effect of pitchfork bifurcation of simple orbits on the eigenfunctions has also been reported [10]. In this Letter, we explore further the connection between certain simple classical periodic orbits, their stability and the localization of the quantum wavefunctions scarred by them, using coupled nonlinear oscillators. In such systems, even in the regions of large scale classical chaos, certain simple periodic orbits with short time periods and high stability are known to scar a series of WKB-like states in the spectrum [11] and recently it has been observed that these states affect the eigenvalue spacing distributions [12]. In particular, our numerical evidence for smooth potentials indicates that such states, scarred by simple periodic orbits, are exponentially localized in the space of unperturbed basis states.

In the general analytical framework of scarring as developed by Berry [13], the spectral Wigner function averaged over a small energy scale is shown to be influenced by isolated periodic orbits semiclassically. The stability of the periodic orbits is shown to affect the scar weight and scar amplitude significantly. The analysis remains limited to averaged Wigner functions and is not valid at the points of bifurcations where Gutzwiller's density of states formula breaks down, leading to predictions of either infinite scar weights or scar amplitudes. In this context we will show that gross measures of individual wavefunction localization, such as entropy, are strongly correlated with the periodic orbit's stability oscillations (with a parameter). When the orbit loses stability we will find that the entropy is also low, with the point of bifurcation being approximately the point of a local minimum in the entropy (again as a function of a parameter). We will note that this does not always coincide with the points at which the Berry formula predicts an infinite scar amplitude.

We study smooth Hamiltonian systems of the form,

$$H(x, y, p_x, p_y; \alpha) = \frac{p_x^2}{2\,m} + \frac{p_y^2}{2\,m} + V(x, y; \alpha) \qquad (1)$$

whose classical dynamics can be regular or chaotic depending on the value of the parameter α. The coupled quartic oscillator given by the homogeneous potential $V(x, y; \alpha) = x^4 + y^4 + \alpha x^2 y^2$, is used below with $m = 1/2$. This system is integrable for $\alpha = 0, 2$ and 6 and exhibits increasing chaos as α is increased beyond 6. The presence of a "channel" in this potential leads to trapping of the particle in a motion along the one dimensional channel periodic orbit $(x, y = 0, p_x, p_y = 0)$, which has

a short time-period and interesting stability properties. The stability of the channel periodic orbit as measured by Tr$J(\alpha)$, where $J(\alpha)$ is the monodromy matrix, displays bounded oscillations as a function of α and for this simple periodic orbit of the Hamiltonian in Eq. (1) the analytical expression for Tr$J(\alpha)$, due to Yoshida [14], is given by,

$$\mathrm{Tr}\ J(\alpha) = 2\sqrt{2}\cos(\frac{\pi}{4}\sqrt{1+4\alpha}), \tag{2}$$

The monodromy matrix refers to the "half Poincaré map" [5], which is the usual Poincaré map without the restriction of positive momentum on the intersection. We note that this is the appropriate monodromy matrix in the semiclassical analysis as well due to the fact that we are restricting the quantum spectrum to the A_1 irreducible representation of the C_{4v} point group which the Hamiltonian possesses. The important inference from Eq. (2) is that this channel orbit changes stability when $\alpha = n(n+1)$, where n is an integer, through a pitchfork bifurcation. Another interesting property we have recently found is that the Poincaré section around channel periodic orbit locally scales with respect to the parameter α for all one-parameter homogeneous two-dimensional Hamiltonian systems, which includes the coupled quartic oscillator and the scaling exponents depend simply on the degree of homogeneity of the potential [15]. In this letter, we will focus our attention on this channel periodic orbit and its influence on the quantum wavefunctions.

The quantization of the Hamiltonian in Eq. (1) is by numerically solving the Schröedinger equation in the basis of the eigenfunctions of the corresponding unperturbed problem, namely the $\alpha = 0$ case in Hamiltonian Eq. (1). The actual basis states $\psi(x, y)$ employed are the symmetrised linear combination of the wavefunctions for the unperturbed problem [16]. The wavefunction for the mth state is given by $\Psi_m(x, y; \alpha) = \sum_{i=1}^{N} a_{m,j}\psi(x, y)$, where j represents the pair of even integers (n_1, n_2) corresponding to the quantum numbers of two one-dimensional states that makes up the basis state and $a_{m,j}$ are the expansion coefficients. The information entropy measure, for the mth state with M components, where M is the dimension of the Hamiltonian matrix, defined as,

$$\mathcal{S}_m^{\alpha} = -\sum_{j=1}^{M} \mid a_{m,j}^{\alpha} \mid^2 \log \mid a_{m,j}^{\alpha} \mid^2 \tag{3}$$

is applied to the wavefunction spectrum and we have shown that the localized states scarred by the channel periodic orbit are identified by a pronounced dip in the information entropy curve and they were also visually identified [11].

The Berry-Voros conjecture that the Wigner function of a typical eigenstate of a quantized chaotic system is a microcanonical distribution on the energy shell is of course flagrantly violated by many scarred states and especially so the channel localized ones. In [13] the conjecture is shown to be true semiclassically when there is a large energy averaging done, while individual isolated orbits enter as density enhancements as fewer eigenstates are averaged over in the spectral Wigner function. The channel orbits are in fact so localized that, as we discuss below, they are in a sense even exponentially localized.

In Fig. 1(a) we show the $a_{m,j}$ plotted as a function of j for one such highly excited state at $\alpha = 90.0$. In Fig. 1(b) $a_{m,j}$ corresponding to a channel localized state at the same value of α is plotted. We immediately recognize that for the channel localized state very few basis states contribute to building up of the wavefunction, and we identify the principal contribution to be coming from (N,0), (N,2), (N,4)... type of basis states, where the even integer N refers to the number of quanta of excitation for the motion along the channel. We observed that the pattern in Fig. 1(b) is qualitatively generic for all the channel localized states. For instance, in Fig. 2 we have the wavefunctions of localized states for the parameters $\alpha = 88.0$ and 96.0.

Significantly, the quanta N also enters an adiabatic formula, with constants $b_0 \approx b_1/3$ and b_1,

$$E_N(\alpha) = b_0\sqrt{\alpha}\left(N + \frac{1}{2}\right)^{1/3} + b_1\left(N + \frac{1}{2}\right)^{4/3} \tag{4}$$

to estimate the eigenvalues E_N of the channel localized states. The existence of such a formula for a subclass of states in a chaotic spectrum is indicative of the special nature of these states as no systematics has yet been uncovered for scarring. Although this formula does not include the stability of the orbit as a parameter we have found that this is most accurate when the orbit is about to lose stability and the wavefunction is highly localized as noted below. We derived Eq. (4) as a consequence of adiabaticity as developed in [17].

Our numerical results show that the channel localized states in the unperturbed basis are dominated by exponentially falling peaks in the quantum number of the motion perpendicular to the direction of the channel; and further that the degree of localization is related to the stability of the channel periodic orbit. In Fig. 3 we plot $\log(a_{m,j(N,n_2)})$ as a function of n_2 for the principal peaks corresponding to the states shown in Fig. 1(b) and Fig. 2 and the fairly good straight lines obtained show that the fall is indeed exponential. The next dominant peak with contributions coming mainly from $(N+2, 0), (N+2, 2), (N+2, 4)$.... basis states also provides an evidence (not shown here) of exponential localization. However, for the third peak, corresponding to $(N + 4, 0)$, the values of $a_{m,j}$ fall within our accuracy of our calculation and hence although unequivocal conclusions cannot be drawn we expect the fall to be exponential.

It has been established that the eigenstates of time-dependent systems, like the kicked rotor, are exponentially localized but the above result is the first observation of exponential localization in smooth chaotic systems. Why not exponential localization for other scarred states? We believe that the answer lies in the fact that (a) the basis in which there is exponential localization belongs to the Hamiltonian, namely Eq. (1) with $\alpha = 0$, for which the scarring orbit is also a valid orbit, (b) The stability of such an orbit is high, the channel orbit never becomes very unstable, however large the nonlinearity. For instance the 45^o straight line periodic orbits could be exponentially localized in the 45^o rotated unperturbed basis, but this orbit becomes extremely unstable thereby creating complex states in which other orbits contribute to the scarred state.

The channel periodic orbit loses stability through a pitchfork bifurcation (Tr $J(\alpha) = -2$) at $\alpha = 90.0$ while giving birth to two other stable orbits. We notice from Fig. 3 that we get the best exponential behavior at $\alpha = 90.0$ and it progressively moves away from exponential behavior as we explore the parameter regions in which the channel orbit also becomes progressively unstable. In Fig. 4 we have striking visual evidence for two wavefunctions at two different α values at which the channel orbit is stable and unstable. The wavefunction at $\alpha = 90.0$ is compact and has almost collapsed on to the periodic orbit in comparison with the wavefunction at $\alpha = 96.0$.

We calculated the average information entropy for a particular α by taking the mean of information entropies, $< S_\alpha >= \sum_\sigma \mathcal{S}^\alpha_\sigma / k$ for a group of k localized states represented by σ within some energy range. The plot of $< S_\alpha >$ in Fig. 5 shows that even this averaged measure reflects the trend observed in the stability oscillations of the channel orbit in the vicinity of $\alpha = 90.0$; although the exact minimum of the entropy seems to be slightly removed from this point of bifurcation. Further evidence of entropy oscillations is seen in the perturbed oscillator system we present below.

In order to check the validity of our finding on other similar systems, we studied the perturbed oscillator, given by the potential $V(x,y) = x^2/2 + y^2/2 + \beta x^2 y^2$ where β is the parameter; we have taken $m = 1$ and $\beta = 0.1$ below. This is a non-homogeneous system and the scaled parameter is $\epsilon = \beta E$, where E is the energy of the system. The advantage of non-homogeneity in this system is that now the stability oscillations of the channel orbit is with respect to ϵ, and hence with a fixed value of β we can study how the wavefunction localization is affected by these oscillations.

The results presented below pertain to the perturbed oscillator, unless otherwise specified. The structure of channel localized wavefunctions in the unperturbed space does have the generic pattern similar to the Fig. 1(b) for the case of coupled quartic oscillator. The principal peak for the three states, with different scaled parameters $\epsilon = 8.03, 19.73$ and 10.65 corresponding to stable and unstable motion of the channel orbit, falls exponentially as is evident from Fig. 6, showing the plot of $\log(a_{m,j})$ against j. Since the scaled parameter ϵ is a function of the energy E, a global picture of the stability of the channel orbit and its influence on the degree of localization, as measured by the information entropy, can be obtained in this system. In Fig. 7 the quantity $|\mathrm{Tr}\ J(\epsilon) - 2|$, an indicator of the stability of the channel orbit, and the information entropy for the first 2000 states are plotted against ϵ (here the monodromy matrix is for the full Poincaré map). Neglecting the envelope of the information entropy, which follows the predictions of random matrix theory [11], the entropy of channel localized states show remarkable oscillations that strongly correlate with the stability oscillations of the channel orbit.

We note from Fig. 7 that the dark circles corresponding to Tr $J(\epsilon) = 2$ are points of pitchfork bifurcation at which channel orbit loses stability, and this correlates strongly with the entropy minima of the channel states. We may note that when the orbit gains stability the entropy is not a minimum although Gutzwiller's trace formula breaks down here as well; and Berry's scarring amplitude formula [13] may diverge. Thus the entropy minima must also have to do with the local structure around the periodic orbit and not depend only on the stability matrix of the scarring orbit.

The study of the simplest scarred states, the channel localized ones, have thus shown interesting exponential localization properties as well as strong correlation of the degree of localization with the local structure of the scarring orbit including its stability. Other states that share some of these properties are being actively studied. The scarred state noted in the experiment in ref. [4] also has been shown to obey an underlying approximate WKB like eigenvalue formula [4], and it may well be possible to experimentally observe some of the phenomena noted in this Letter; either by such experiments or possible microwave cavity experiments [18].

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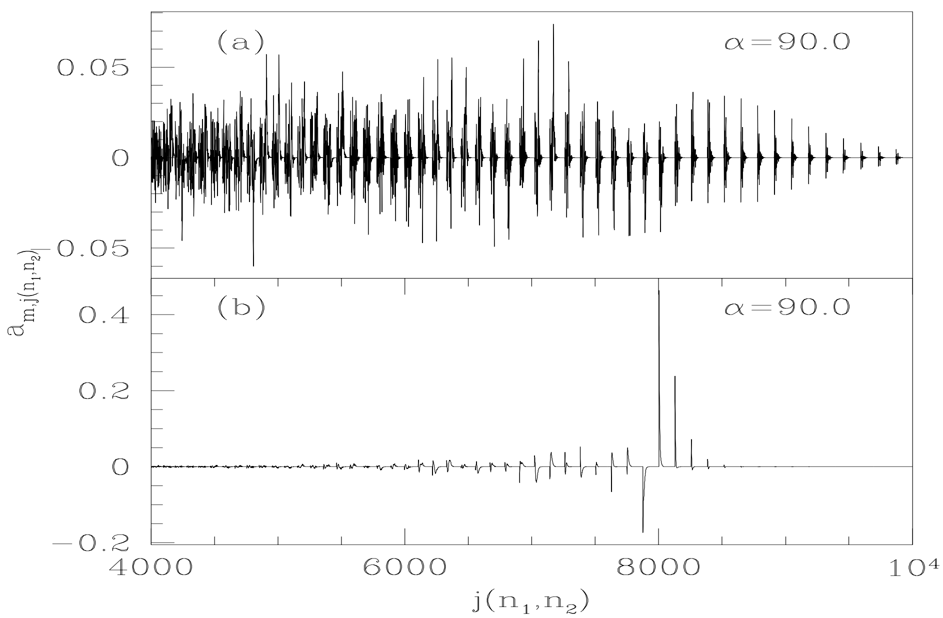

FIG. 1. Significant expansion coefficients of two states from the same spectrum for the quartic oscillator; (a) a typical state, number 1971 from the ground state and (b) state number 1774, a channel localized state.

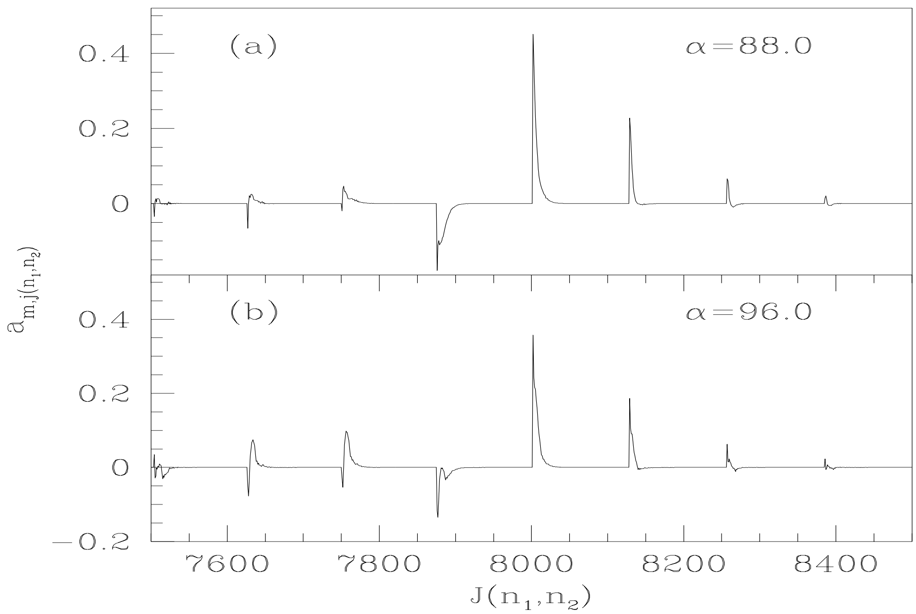

FIG. 2. (a) Significant expansion coefficients of two channel localized states at different parameter values, (a) state number 1786 and (b) state number 1740. The principal peak $N = 252$ (see text) for these states and those in Fig. 3.

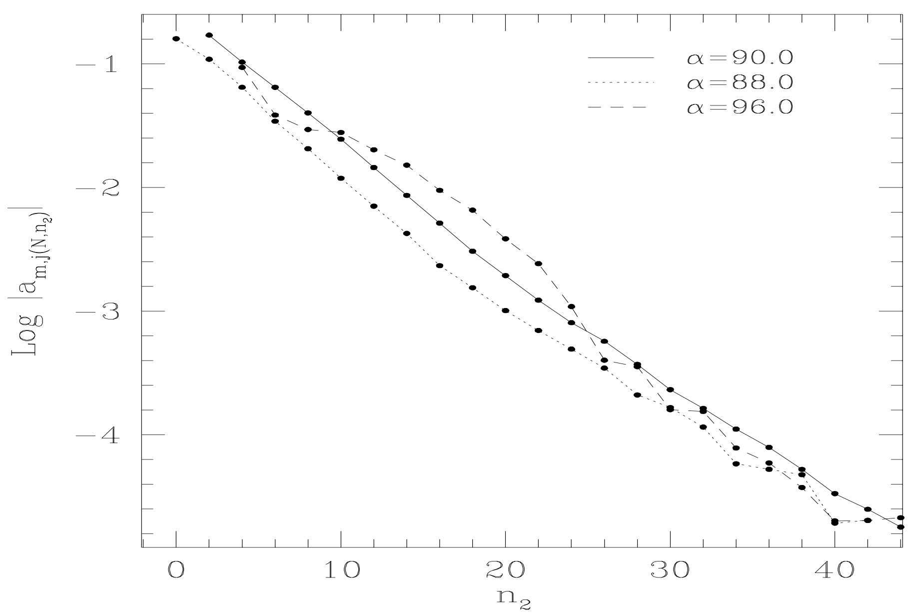

FIG. 3. Exponential localization in states 1786, 1774 and 1740 at $\alpha = 88.0, 90.0$ and 96.0 respectively. The values of n_2 for each curve is shifted for clarity.

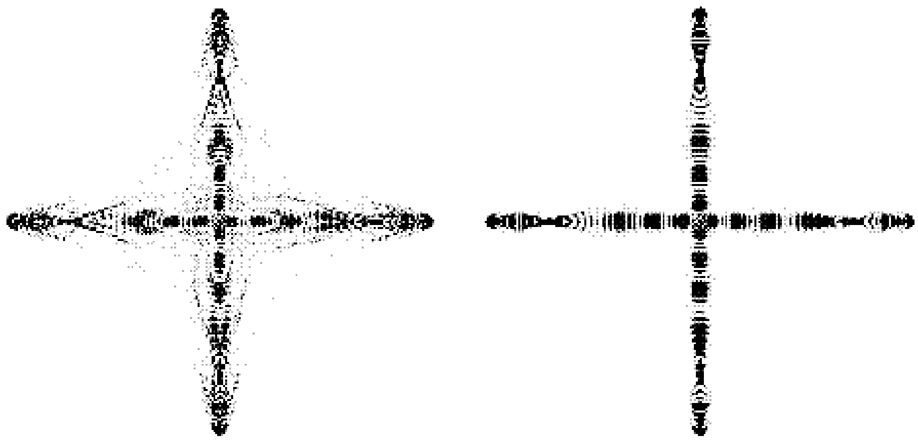

FIG. 4. Configuration space channel localized wavefunction densities for states 1740 (α = 96.0, left) and 1774 (α = 90.0, right)
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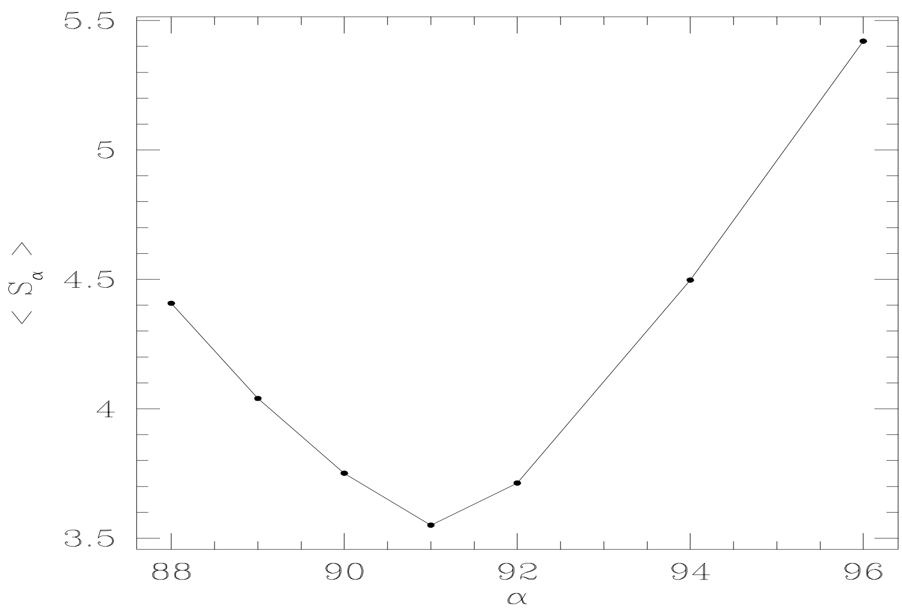

FIG. 5. Averaged information entropy plotted against α for the quartic oscillator.

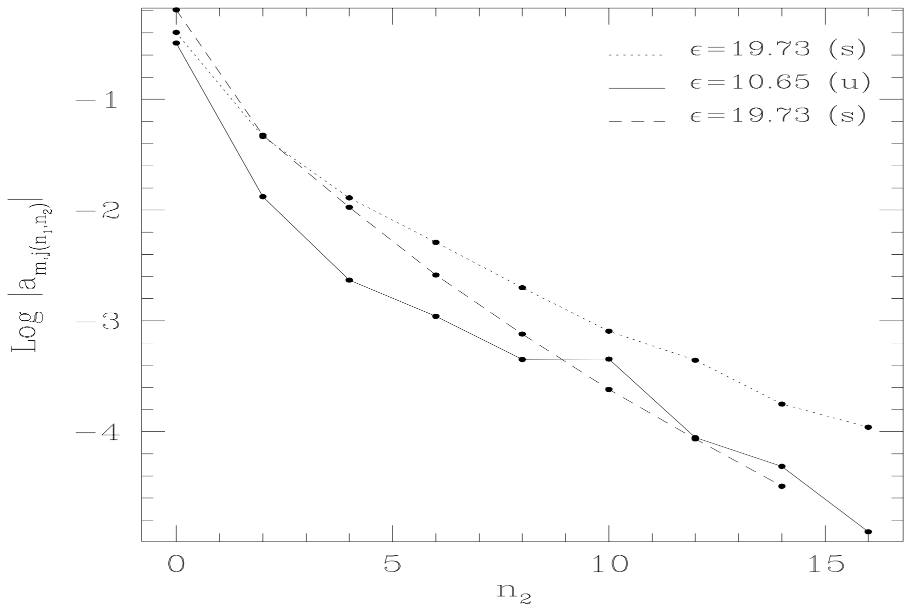

FIG. 6. Exponential localization for three channel localized states of the perturbed oscillator. u and s represent unstable and stable channel orbits. n_1 for the three curves above are not the same.

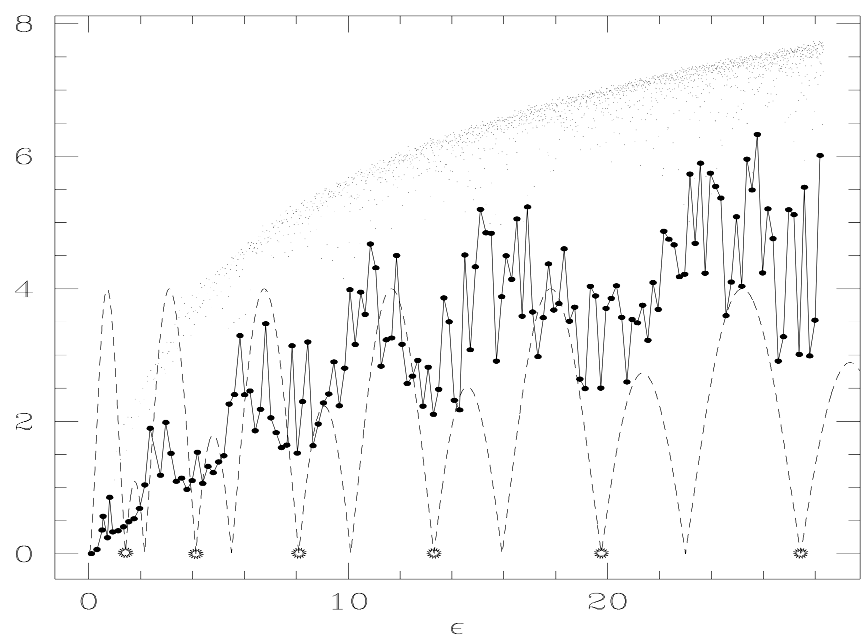

FIG. 7. Entropy of the first 2000 states of the perturbed oscillator. The solid line connects the channel localized states only while the dashed line is $|\mathrm{Tr}(J(\epsilon)) - 2|$. See text for details.